

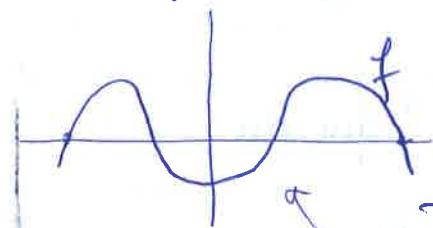
Lecture 28:

(1)

from now on, we will only be working with the continuous Fourier transform \hat{f} of any $f: \mathbb{R} \rightarrow \mathbb{C}$.
for that reason, we return to the original notation f .

The uncertainty principle for any $f: \mathbb{R} \rightarrow \mathbb{R}$ and its Fourier transform can also be demonstrated through this basic, yet very important, property of the Fourier transform:

Let $f: \mathbb{R} \rightarrow \mathbb{C}$ for any scalar $\alpha \in \mathbb{R}$, we



consider $f_\alpha: \mathbb{R} \rightarrow \mathbb{C}$
 $f_\alpha(x) = f(\alpha x),$
 $x \in \mathbb{R}$.

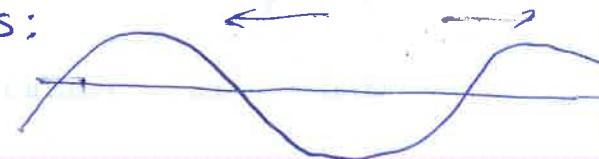
When, for instance, $\alpha=100$, then the graph of f_α is just a dilating the domain of f .

is just a squeezed version of the graph of f

inwards



When $\alpha=\frac{1}{100}$, then the graph of f_α comes from stretching the graph of f outwards:



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So, the larger λ gets, the more localised f becomes.

And:

$$\hat{f}_\lambda(x) = \frac{1}{\lambda} \cdot \hat{f}\left(\frac{x}{\lambda}\right), \quad \forall x \in \mathbb{R}:$$

$$\begin{aligned} \hat{f}_\lambda(x) &= \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ixy} f_\lambda(y) dy = \\ &= \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ixy} f(\lambda y) dy \quad \begin{array}{l} \lambda y = u \Rightarrow y = \frac{u}{\lambda} \\ dy = \frac{du}{\lambda} \end{array} \\ &= \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ix \cdot \frac{u}{\lambda}} f(u) \frac{du}{\lambda} = \\ &= \frac{1}{\lambda} \cdot \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{i\left(\frac{x}{\lambda}\right)u} f(u) du = \\ &= \frac{1}{\lambda} \cdot \hat{f}\left(\frac{x}{\lambda}\right), \quad \forall x \in \mathbb{R}. \end{aligned}$$

Notice that, the larger λ gets, the more squeezed the graph of f becomes, thus f becomes a more localised function f_λ ; and the Fourier transform of this more localised

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version \hat{f}_x is a more stretched out version of the Fourier transform of f .

Thus, the more we localise f , the more we spread the Fourier transform.

The Schrödinger equation:

Let q be a quantum particle.

Let $X(t)$ be the random variable, that is the position of q at time t .

Suppose that we know the probability density $|f|^2$ of $X(0)$, i.e. of the position at time 0.

In other words, suppose that we know that the probability $P(a \leq X(0) \leq b)$ that q is between a and b at time 0 is



this implies that $\int_a^b |f(x)|^2 dx = 1$,
as f has to be somewhere at time $t=0$.

$$P(a \leq X(0) \leq b) = \int_a^b |f(x)|^2 dx, \text{ if } a < b \text{ in } \mathbb{R}.$$

Then, what is the probability density $|u(x, t)|^2$ of the position $X(t)$ of q at any time t ?

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$u(x, t)$, $\forall x \in \mathbb{R}, t \in \mathbb{R}$, is given by the Schrödinger equation (which makes sense even when $\int_{-\infty}^{+\infty} |f|^2 \neq 1$):

$$\Delta_x u(x, t) = -i \frac{\partial}{\partial t} u(x, t),$$

where $u(x, 0) = f(x) \quad \forall x \in \mathbb{R}$, for some $f \in L^2(\mathbb{R})$.

We solve this using the basic property of the Fourier transform that it turns differentiation into multiplication:

Let $f \in L^1(\mathbb{R})$, $f' \in L^1(\mathbb{R})$, f' continuous.

this implies that the area under $|f|$ is finite, so $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$

so that \hat{f} , \hat{f}' are well-defined

so that f' can participate in integration by parts

Then,

$$\hat{f}'(x) = (\hat{i}x) \cdot \hat{f}(x), \quad \forall x \in \mathbb{R}.$$

Proof: $\hat{f}(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ixy} f(y) dy =$

$$= \frac{1}{2\pi} \cdot [e^{-ixy} f(y)]_{y=-\infty}^{+\infty} - \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} (e^{-ixy})' f(y) dy$$

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$$\begin{aligned}
 &= \frac{1}{2n} \cdot \left(\lim_{y \rightarrow +\infty} e^{-ixy} f(y) - \lim_{y \rightarrow -\infty} e^{-ixy} f(y) \right) - \frac{(-i)}{2n} \int_{-\infty}^{+\infty} e^{-ixy} f(y) dy = \\
 &= (ix) \cdot \frac{1}{2n} \int_{-\infty}^{+\infty} e^{-ixy} f(y) dy = (ix) \cdot \hat{f}(x), \quad \forall x \in \mathbb{R}.
 \end{aligned}$$

Applying this to the Schrödinger equation, we get

$$\begin{aligned}
 \widehat{\Delta_x u}(x,t) &= -i \widehat{\frac{\partial}{\partial t} u}(x,t) \quad \left(\begin{array}{l} \text{the Fourier transform} \\ \text{is taken w.r.t. } x \end{array} \right) \\
 \text{or} \\
 (ix)^2 \widehat{u}(x,t) &= -i \cdot \frac{1}{2n} \cdot \int_{-\infty}^{+\infty} e^{-ixy} \frac{\partial}{\partial t} u(x,t) dx = \\
 &= -i \cdot \frac{\partial}{\partial t} \left(\frac{1}{2n} \int_{-\infty}^{+\infty} e^{-ixy} u(x, \frac{t}{n}) dx \right) = \\
 &= -i \frac{\partial}{\partial t} \widehat{u}(x,t).
 \end{aligned}$$

$$So, \quad \frac{\partial}{\partial t} \widehat{u}(x,t) = \frac{x^2}{i} \widehat{u}(x,t) = -ix^2 \widehat{u}(x,t)$$

$$\Rightarrow \widehat{u}(x,t) = e^{-ix^2 t} c(x) \quad \forall x \in \mathbb{R}, t \in \mathbb{R}$$

$$\begin{aligned}
 \text{where } \widehat{u}(x,0) &= e^{-ix^2 \cdot 0} c(x) = c(x) \\
 \Leftrightarrow c(x) &= \widehat{u}(x,0) = \hat{f}(x), \quad \forall x \in \mathbb{R}.
 \end{aligned}$$

$$So, \quad \widehat{u}(x,t) = e^{-ix^2 t} \hat{f}(x), \quad \forall x \in \mathbb{R}.$$

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Since we know $\hat{u}(x,t)$, we also know $u(x,t)$, by the Fourier inversion formula:

$$u(x,t) = \int_{-\infty}^{+\infty} e^{ixy} \hat{u}(y,t) dy = \\ = \int_{-\infty}^{+\infty} e^{ixy} \cdot (e^{-iy^2 t} \hat{f}(y)) dy,$$

$x \in \mathbb{R}$, $t \in \mathbb{R}$.

This technique of using the Fourier transform is very useful in solving PDE in general.

Notice that the Schrödinger flow preserves energy:

$$\forall t \in \mathbb{R}, \int_{-\infty}^{+\infty} |u(x,t)|^2 dx \stackrel{\text{Plancherel}}{=}$$

$$= 2\pi \cdot \int_{-\infty}^{+\infty} |\hat{u}(x,t)|^2 dx \stackrel{\hat{u}(x,t) = e^{-ix^2 t} \hat{f}(x) \quad \forall x \in \mathbb{R}}{=}$$

$$= 2\pi \cdot \int_{-\infty}^{+\infty} |e^{-ix^2 t} \cdot \hat{f}(x)|^2 dx = 2\pi \cdot \int_{-\infty}^{+\infty} |\hat{f}(x)|^2 dx$$

unit vector

Plancherel $\int_{-\infty}^{+\infty} |f(x)|^2 dx$

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Thus: the energy of the solution at any time t is the energy of the initial data.

In the particular case where $\int |f|^2 = 1$,

where we have the quantum particle interpretation

we described earlier, then the energy preservation is expected:

$\int_{-\infty}^{+\infty} |u(x,t)|^2 dx = \text{the probability the quantum particle is somewhere at time } t = 1,$

no matter what t is.

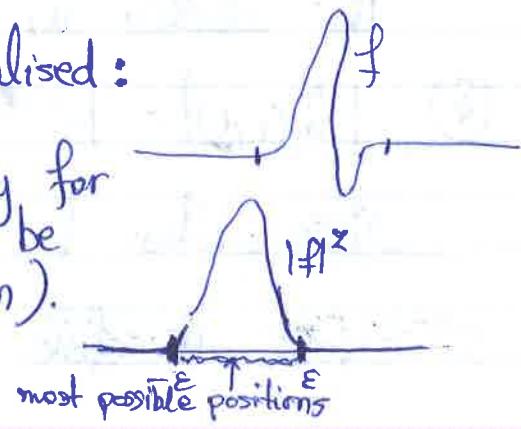
→ Connection of the Schrödinger equation with the uncertainty principle:

The momentum of the quantum particle q at time t is any x s.t. $u(x,t)$ is significant. (really a constant multiple of such x .)

Now, suppose that we know with some accuracy the position of q at time 0,

this means that f is highly localised:

(Indeed, that is the only way for the probability density $|f|^2$ to be highly localised around one position).



So, we know by the uncertainty principle that f is not localised, i.e. there are many $\hat{u}(x, 0)$ s.t. $\hat{u}(x, 0)$ is non-negligible, i.e. significant.

So, the momentum of q at time 0 cannot be specified.

One can also see this from the discrete perspective:
 Writing f as a Fourier series, there are many exponentials $f(k) e^{ikx}$ that contribute to f , thus many k 's s.t. $f(k)$ non-negligible: so none of these k 's qualifies as a best candidate to be the momentum of f .

The continuous Fourier transform, which lead to the solution u , gives some more insight: $\forall x \in \mathbb{R}, \forall t \in \mathbb{R}$

$$(*) |\hat{u}(x, t)| = |e^{-i x k t} \hat{f}(k)| = |\hat{f}(k)| \quad \forall x \in \mathbb{R},$$

When f is localised, \hat{f} is not, thus, by $(*)$,

nor is $\hat{u}(\cdot, t)$, i.e. $\hat{u}(x, t)$ non-negligible for many $x \in \mathbb{R}$.

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So, when we know with some accuracy the position
of q at time 0,

then we cannot specify its momentum
at any time t .